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Introduction to Special Issue: Dedekind and the Philosophy of Mathematics

Richard Dedekind (1831–1916) was a contemporary of Bernhard Riemann, Georg Cantor, and Gottlob Frege, among others. Together, they revolutionized mathematics and logic in the second half of the nineteenth century. Dedekind had an especially strong influence on David Hilbert, Ernst Zermelo, Emmy Noether, and Nicolas Bourbaki, who completed that revolution in the twentieth century. With respect to mainstream mathematics, he is best known for his contributions to algebra and number theory (his theory of ideals, the notions of algebraic number, field, module, *etc.*). With respect to logic and the foundations of mathematics, many of his technical results — his conceptualization of the natural and real numbers (the Dedekind-Peano axioms, Dedekind cuts, *etc.*), his analysis of proofs by mathematical induction and definitions by recursion (extended to the transfinite by Zermelo, John von Neumann, *etc.*), his definition of infinity for sets (Dedekind-infinite), *etc.* — have been built into the very fabric of twentieth- and twenty-first-century set theory, model theory, and recursion theory. And with some of his methodological innovations he even pointed towards category theory. (*Cf.* [Ferreirós, 1999; Corry, 2004; Reck, 2016] also for further references.)

No philosopher of mathematics today can afford to be ignorant of Dedekind's technical results. His more philosophical views, as well as other philosophical aspects of his mathematical style, have received much less attention, however, at least until recently. To some degree, this is due to the fact that he did not elaborate much on these views and aspects, especially compared to figures such as Cantor, Frege, Bertrand Russell, or Henri Poincaré. Partly it also results from early philosophical criticisms by Frege and Russell, later re-emphasized by Michael Dummett, George Boolos, and others [Reck, 2013b]. Besides Russell's antinomy, which applied to Dedekind's logical framework as much as to Frege's, it was mainly the charge of psychologism that appeared to disqualify him as a first-rate philosopher, particularly among Fregeans. But the structuralist aspect of his position was also received unsympathetically for a while (with exceptions, *cf.* [Cassirer, 1907; 1910]). In contrast, three recent developments have started to bring him back to the attention of philosophers of mathematics: the revival of structuralist conceptions of mathematical objects (Benacerraf, Resnik, Shapiro, Hellman, Parsons, *etc.*), frequently claimed to be in line with, or at least in the spirit of, Dedekind; increased interest in the philosophy of mathematical practice, including in what is sometimes called 'methodological structuralism' [Reck and Price, 2000]; and the reconsideration of logicism, both from a historical point of view, with Dedekind being recognized as an early proponent [Reck, 2013a], and from a systematic perspective, including the use of abstraction principles [Linnebo and Pettigrew, 2014].

This special issue of *Philosophia Mathematica* is meant to reflect, as well as to reinforce, the revival of Dedekind's views and Dedekindian themes in the philosophy of mathematics. It consists of four new papers (previously available only in online-first versions). Two of them focus on mathematical practice and methodology, as seen from a philosophical perspective (those by Sieg and Schlimm and by Ferreirós). The other two concern more traditional philosophical themes, namely logicism and the objectivity of mathematics (Klev and Yap, respectively). As the following summaries will make evident, there are various interconnections between these pieces.

DEDEKIND, METHODOLOGICAL STRUCTURALISM, AND THE NOTION OF MAPPING

In their co-written paper, 'Dedekind's abstract concepts: Models and mappings', Wilfried Sieg and Dirk Schlimm develop a distinctive, historically grounded, and philosophically rich account of the sense in which Dedekind's writings embodied an original and revolutionary structuralist methodology. (The paper is a continuation of [Sieg and Schlimm, 2005].) The lens through which they see him is the 'axiomatic standpoint', as developed further by Hilbert and Noether later, and involving three aspects especially: abstract concepts (continuity, simple infinity, field, *etc.*), models (subfields of the complex numbers, models of the Dedekind-Peano Axioms), and mappings (the successor function, algebraic morphisms). Particular attention is paid to the emergence of the notion of structure-preserving mapping, from a few germs in Dedekind's predecessors, especially Gauss and Dirichlet, to their blossoming in his own work. In the latter, this development went through several forms or stages, from Dedekind's early work in group theory (during the 1850s) through his celebrated work in algebraic number theory and initial drafts of his foundational writings (1860s–70s) to his mature writings (from the mid-1880s on). There is a related shift in Dedekind's terminology (from 'substitutions' through 'permutations' and 'correspondences' to 'mappings'), traced carefully by Sieg and Schlimm as well.

The focus in Sieg's and Schlimm's paper is on the emergence of structuralist mathematics in Dedekind's writings up to the 1880s. In José Ferreirós's paper, 'Dedekind's map-theoretic period', that emergence is acknowledged too, but then an original twist is added. Ferreirós argues that there is a subtle, gradual shift in Dedekind's writings, from a primarily set-theoretic orientation, from 1858 to 1887, to a more map-theoretic focus, most prominently in his writings from 1887 to 1894. Like Sieg and Schlimm, Ferreirós discusses Dedekind's foundational writings in this connection, especially *Was sind und was sollen die Zahlen*? (an 1887 draft of it, as well as the final version from 1888). But two less familiar contributions receive careful attention as well: Dedekind's strikingly modern treatment of Galois theory, integrated into his algebraic number theory, in which the notion of structure-preserving mapping is placed at the very center (transforming Galois theory from the investigations of equations and their solutions to the study of field extensions and corresponding automorphisms); and his introduction of a radically new, distinctively map-theoretic, version of the continuum in an unfinished fragment from 1891 (pointing towards Baire space). The paper concludes with a discussion of how this gradual shift in Dedekind's orientation fits with his logicism.

DEDEKIND, PHILOSOPHY OF MATHEMATICS, AND THE HISTORY OF PHILOSOPHY

In the secondary literature, Dedekind is sometimes mentioned as a major early logicist, besides Frege and Russell. Indeed, Dedekind himself talks about his goal of establishing that arithmetic is 'a part of logic', or of providing a 'purely logical construction' for the natural and real numbers, especially in the Preface to his 1888 essay. However, he is much less explicit than Frege or Russell about the logical framework assumed in the background. Moreover, there are some open questions about his basic concepts, especially those of set and function, and their logical nature. In Ansten Klev's paper, 'Dedekind's Logicism', such issues are addressed directly. (In some respects, this paper is a continuation of [Klev, 2011].) After some initial clarifications concerning the notion of logicism operative in Dedekind's writings, developed with reference to Rudolf Carnap's influential characterization of it, Klev focuses on Dedekind's claim that functional thinking is 'indispensible for human thought'. His core suggestion is to conceive of that claim along Kantian lines, and more specifically, to see the ability to think 'functionally' as intrinsically tied to 'the understanding' in Kant's sense. Implicitly Dedekind's use of 'logic' is thus put in the context of 'transcendental logic', as opposed to more recent alternatives, *e.g.*, Frege's and Alfred Tarski's, that are also considered briefly.

Finally, there is Audrey Yap's paper, 'Dedekind and Cassirer on mathematical concept formation'. In it, Dedekind's logicism, his alleged psychologism, and connections to Kant are addressed once again, but in a different way, namely by connecting them to the neo-Kantian philosopher Ernst Cassirer. While Cassirer's sympathetic reception of Dedekind's philosophical views has been noted more generally (*cf.* [Friedman, 2000; Heis, 2010; Reck, 2013b], Yap focuses specifically on the accounts of subjectivity and objectivity with respect to mathematical concept formation in Cassirer's book *Substance and Function*. She suggests that Cassirer's 'function-based' perspective on mathematics, together with his juxtaposition of two kinds of abstraction, provides a helpful framework for re-evaluating Dedekind's position. More particularly, it allows for the rejection of a naïve psychologistic reading of him, as presented prominently in Dummett's work; and it does so in a more nuanced way than earlier defenses of Dedekind by W.W. Tait and David McCarty. As a result, we can take his language of 'mental creation' seriously after all, namely along the lines of the transcendental psychology specific to Cassirer's Neo-Kantianism. These suggestions are used, moreover, to reinforce a reading of Dedekind's position as 'logical structuralism' (*cf.* [Yap, 2009a;b; Reck, 2003]).

CONCLUDING REMARKS

Comparing all four papers, the following related suggestions concerning Dedekind's philosophical views are perhaps most noteworthy. First, it is not just the notion of set that was crucial for Dedekind's logicism, thus for his revolutionary rethinking of the pure mathematics of his time, but also, and perhaps more, the notion of mapping or function. Second, with respect to mathematical practice it is Dedekind's substantive and self-conscious use of structure-preserving mappings (morphisms, including homomorphisms and isomorphisms) that constitutes his most important contribution. Third, Dedekind's emphasis on 'functional thinking' allows for some illuminating comparisons to Kantian ideas, including Kant's transcendental logic, but also, perhaps more appropriately in the end, the Neo-Kantianism of Ernst Cassirer. And fourth, such comparisons shed new light on the notion of 'logic' involved in Dedekind's writings.

It is not to be expected that the articles in this special issue are the last word on Dedekind's philosophical views, not even on those aspects directly addressed in them: his structuralism, his logicism, and his more general methodology. In fact, an ongoing debate in the literature about how exactly to understand Dedekind's structuralism should be mentioned here, one that forms the background for some of the present papers (*cf.* the interpretation in [Sieg and Schlimm, 2005], their paper in this issue, as well as [Sieg and Morris, forthcoming], on the one hand, and the works by Reck and Yap already mentioned, on the other hand). There is also an emerging debate about Dedekind's logicism, including a recent challenge to categorizing Dedekind that way at all, given some important differences to Frege's and Russell's versions of it (*cf.* the suggestions in [Benis-Sinaceur *et al.*, 2015] versus the pieces by Ferreirós, Klev, and Reck in the references below). Finally, much more can, and should, be said about philosophically relevant aspects of Dedekind's methodology (compare, *e.g.*, [Avigad, 2006; Detlefsen, 2011; Haffner, 2014]). Then again, each of the four articles included here contributes to the corresponding debates in novel and substantive ways, thus helping to raise the level of philosophical discussions about Dedekind significantly.

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